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LINEAR ANALYSIS OF WIND INDUCED BAROCLINIC FLOW IN LONG NARROW BASINS

by

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1. INTRODUCTION

The summer density distribution in many lakes is characterized by the existence of a thermocline, i.e. a region of rapid temperature change separating a relatively warm, light surface layer from the relatively cold, heavy bottom layer. According to Csanady (1975) the steepest temperature gradient usually occurs at a depth of about 20 meters while the density difference between the top and the bottom layer is of the order of one part in one thousand.

We consider the linearized equation of motion for a stratified fluid (two-layers) and pay particular attention to baroclinic motion of a frictionless interface occuring at the presumed location of the depth with the greatest temperature gradient. The main cause for motion of the thermocline in most lakes is the wind stress at the lake surface. The approach taken in this report is to relate the depth integrated momentum of each layer separately to pressure gradients, Coriolis force and wind stress. The vertical equation of motion is replaced by the hydrostatic approximation and we are left with two horizontal momentum equations and one continuity equation for each layer.

For the lower layer with wind stress only acting along the major axis of the lake, we obtain

$$\frac{\partial U}{\partial t} - fV = c^2 \frac{\partial \zeta}{\partial x} + \frac{\tau_x}{\rho(1+h/H)} \tag{1-1}$$

$$\frac{\partial V}{\partial t} + fU = c^2 \frac{\partial \zeta}{\partial y} \tag{1-2}$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \tag{1-3}$$

In the above equations x is the coordinate along the major axis of the basin and y the coordinate along the minor axis, and t is time. The origin of the coordinate system is located at the midpoint of one end of the basin as shown in Figure 1. The symbols are defined in Table 1.

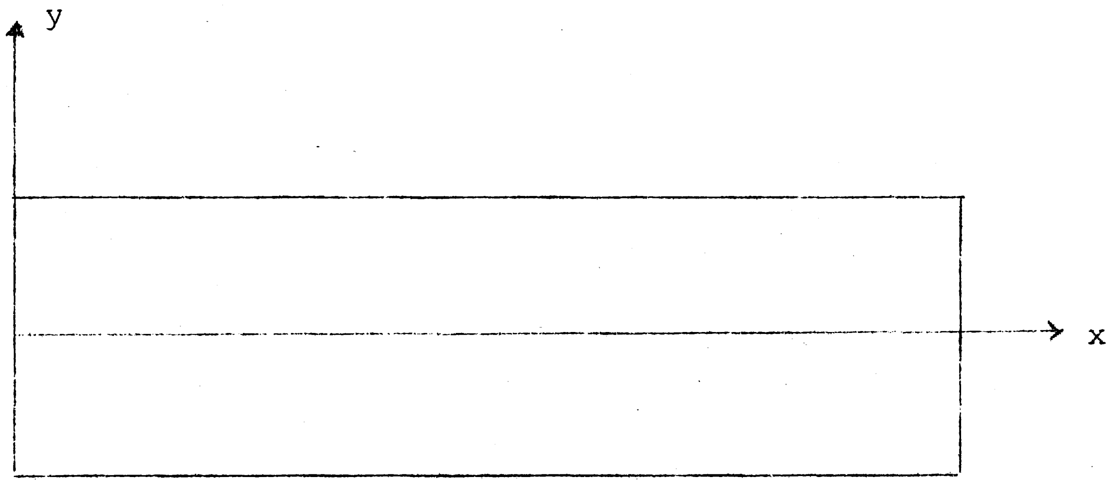

Figure 1 Basin Geometry

f = Coriolis parameter

τ_x = stress in the direction of the x-axis

ζ = elevation of the thermocline above equilibrium

g = gravitational attraction

$\begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$ component of the velocity integrated over the bottom layer (i.e. transport)

$\begin{Bmatrix} \ell \\ b \end{Bmatrix} = \begin{Bmatrix} \text{length} \\ \text{width} \end{Bmatrix}$ of the basin

$\begin{Bmatrix} h \\ H \end{Bmatrix}$ = height of the $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ layer

$\begin{Bmatrix} \rho \\ \rho+\Delta\rho \end{Bmatrix}$ = density of the $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ layer

$c^2 = g\Delta\rho h/(\rho(1+h/H))$

Table 1 : Definition of symbols

For a detailed derivation of these basic equations, see Sverdrup (1957) or Proudman (1953). Two excellent review articles on lakes by Csanady occur in Csanady (1975) and Lerman (1978).

The specific problem to be considered in this report is the determination of the space and time behaviour of the thermocline when a wind stress is applied to the quiescent basin. In developing the solution it is advantageous to introduce dimensionless coordinates X, Y, T by

$$X = 2x/b \;, \qquad Y = 2y/b \;, \qquad T = 2ct/b \tag{1-4}$$

Thus our basic equations may be reweitten as

$$\frac{\partial U}{\partial T} - \xi V = c\,\frac{\partial \zeta}{\partial X} + \frac{b\tau_X}{2c\rho(1+h/H)} \tag{1-5}$$

$$\frac{\partial V}{\partial T} + \xi U = c\,\frac{\partial \zeta}{\partial Y} \tag{1-6}$$

$$c\,\frac{\partial \zeta}{\partial T} = \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \tag{1-7}$$

with $\xi = fb/2c$ equal to the ratio of the half width of the channel to the "internal radius of deformation".

2. SOLUTION FOR AN INFINITE CHANNEL

2.1 Solution for a wind stress of the form $\tau_x = \tau(T)e^{\gamma Y}$

We first note that because of the τ_x term, the linear differential equations are not homogeneous. In order to obtain a homogeneous set of equations, we write

$$\begin{aligned} \zeta(X,Y,T) &= \zeta'(Y,T) + \zeta^2(X,Y,T) \\ U(X,Y,T) &= U'(Y,T) + U^2(X,Y,T) \\ V(X,Y,T) &= V'(Y,T) + V^2(X,Y,T) \end{aligned} \tag{2-1}$$

where ζ', U', V' is the solution for an infinite channel with motion started from rest, i.e.

$$\zeta'(Y,0) = U'(Y,0) = V'(Y,0) = 0 \tag{2-2}$$

We also impose the boundary condition that the lateral component of velocity vanishes at the edges of the channel. This implies that

$$V'(\pm 1,T) = 0 \tag{2-3}$$

With these assumptions the differential equations for ζ', U' and V' become

$$\begin{aligned} \frac{\partial U'}{\partial T} - \xi V' &= \frac{b\tau_x}{2c\rho(1+h/H)} \\ \frac{\partial V'}{\partial T} + \xi U' &= c\,\frac{\partial \zeta'}{\partial Y} \\ c\,\frac{\partial \zeta'}{\partial T} &= \frac{\partial V'}{\partial Y} \end{aligned} \tag{2-4}$$

If the Laplace transform is applied to (2-4) and (2-3) and use is made of the initial condition (2-2) we arrive at the following system for the transformed quantities (denoted by a "bar")

$$s\bar{U}' - \xi\bar{V}' = Q\,e^{\gamma Y} \equiv \frac{b\bar{\tau}}{2c\rho(1+h/H)}\,e^{\gamma Y}$$

$$s\bar{V}' + \xi\bar{U}' = c\,\frac{d\bar{\zeta}'}{dY} \tag{2-5}$$

$$cs\bar{\zeta}' = \frac{d\bar{V}'}{dY}$$

$$\bar{V}'(\pm 1, T) = 0 \tag{2-6}$$

Uncoupling the differential equation in (2-5) gives

$$\frac{d^2\bar{V}'}{dY^2} - \kappa^2\bar{V}' = \xi Q e^{\gamma Y}\,, \quad \kappa^2 = s^2 + \xi^2 \tag{2-7}$$

The solution of (2-7) which satisfies boundary condition (2-6) is

$$\bar{V}' = \frac{\xi Q}{\kappa^2-\gamma^2}\left[\frac{e^{\gamma}\sinh\kappa\,(1+Y) + e^{-\gamma}\sinh\kappa\,(1-Y)}{\sinh 2\kappa} - e^{\gamma Y}\right] \tag{2-8}$$

while from equation (2-5) we obtain

$$\bar{\zeta}' = \frac{\xi Q\kappa}{cs(\kappa^2-\gamma^2)}\left[\frac{e^{\gamma}\cosh\kappa\,(1+Y) - e^{-\gamma}\cosh\kappa\,(1-Y)}{\sinh 2\kappa} - \gamma e^{\gamma Y}\right] \tag{2-9}$$

$$\bar{U}' = \frac{\xi^2 Q}{s(\kappa^2-\gamma^2)}\left[\frac{e^{\gamma}\sinh\kappa\,(1+Y) + e^{-\gamma}\sinh\kappa\,(1-Y)}{\sinh 2\kappa}\right] + \frac{Q(s^2-\gamma^2)}{s(\kappa^2-\gamma^2)}\,e^{\gamma Y} \tag{2-10}$$

If $\gamma = 0$ these reduce to

$$\bar{V}' = \frac{\xi Q}{\kappa^2}\left[\frac{\cosh\kappa Y}{\cosh\kappa} - 1\right] \tag{2-11}$$

$$\bar{\zeta}' = \frac{\xi Q}{cs\,\kappa}\left[\frac{\sinh \kappa Y}{\cosh \kappa}\right] \tag{2-12}$$

$$\bar{U}' = \frac{sQ}{\kappa^2}\left[1 + \frac{\xi^2}{s^2}\,\frac{\cosh \kappa Y}{\cosh \kappa}\right] \tag{2-13}$$

2.1 Solution for $\tau_x = \tau_0 H(T)$, $H(T)$ = Heaviside function

For a wind distribution that is uniform across the lake and at $T = 0$ rises abruptly from rest to a steady value of τ_0, we have $\gamma = 0$ and $\bar{\tau}_x = \tau_0/s$. Thus the solution given by (2-11) - (2-13) takes the simple form

$$\bar{V}' = \frac{\xi c Q''}{s\kappa^2}\left(\frac{\cosh \kappa Y}{\cosh \kappa} - 1\right) \qquad Q'' = \frac{sQ}{c} = \frac{b\tau_0}{2c^2\rho(1+h/H)} \tag{2-14}$$

$$\bar{\zeta}' = \frac{\xi Q''}{s^2\kappa}\,\frac{\sinh \kappa Y}{\cosh \kappa} \tag{2-15}$$

$$\bar{U}' = \frac{cQ''}{s^2\kappa^2}\left(s^2 + \xi^2\,\frac{\cosh \kappa Y}{\cosh \kappa}\right) \tag{2-16}$$

The inverse Laplace transform of the term of particular interest, $\bar{\zeta}'$, can be obtained using residue theory in the following manner. $\bar{\zeta}'$ is analytic except for a pole of order 2 at $s = 0$ and simple poles at $s = \pm i\sqrt{\xi^2+(n\pi/2)^2} = \pm i\, s_n$, n odd.

The residue at $s = 0$ is $\dfrac{T \sinh \xi Y}{\xi \cosh \xi}$, while at $s = \pm i\, s_n$ we obtain

$$\frac{\pm i \sin(n\pi Y/2)\exp(\pm i s_n T)}{\sin(n\pi/2)\; s_n^{\,3}}$$

Since the function is of order $s^{-5/2}$ for $\mathrm{Re}(s) > 0$, the inverse transform is simply the sum of the residues and

$$\zeta'(Y,T) = Q''\left[\frac{T\sinh\xi Y}{\cosh\xi} + 2\xi\sum_{n=1}^{\infty}\frac{(-1)^n \sin(\lambda_{2n-1}Y)\sin(\sqrt{\xi^2+\lambda^2_{2n-1}}T)}{(\xi^2+\lambda^2_{2n-1})^{3/2}}\right] \tag{2-17}$$

where $\lambda_n = n\pi/2$.

This implies

$$V'(Y,T) = cQ''\left[\frac{\cosh\xi Y}{\xi\cosh\xi} - \frac{1}{\xi} - 2\xi\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\lambda_{2n-1}Y)\cos(\sqrt{\xi^2+\lambda^2_{2n-1}}T)}{\lambda_{2n-1}(\xi^2+\lambda^2_{2n-1})}\right. \tag{2-18}$$

$$U'(Y,T) = cQ''\left[\frac{T\cosh\xi Y}{\cosh\xi} - 2\xi^2\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\lambda_{2n-1}Y)\sin(\sqrt{\xi^2+\lambda^2_{2n-1}}T)}{\lambda_{2n-1}(\xi^2+\lambda^2_{2n-1})^{3/2}}\right. \tag{2-19}$$

If we now consider the situation where $\xi < 1$ and expand the functional forms of ζ', U' and V' as power series in ξ, we obtain

$$\zeta'(Y,T) = Q''\xi\left[TY + 2\sum_{n=1}^{\infty}\frac{(-1)^n\sin\lambda_{2n-1}Y\sin\lambda_{2n-1}T}{\lambda^3_{2n-1}} + O(\xi^3)\right]$$

$$V'(Y,T) = Q''\xi c\left[(Y^2-1)/2 - 2\sum_{n=1}^{\infty}\frac{(-1)^n\cos\lambda_{2n-1}Y\cos\lambda_{2n-1}T}{\lambda^3_{2n-1}} + O(\xi^3)\right]$$

$$U'(Y,T) = Q''c\left[T(1+\xi^2(Y^2-1)/2 - 2\xi^2\sum_{n=1}^{\infty}\frac{(-1)^n\cos\lambda_{2n-1}Y\sin\lambda_{2n-1}T}{\lambda^4_{2n-1}} + O(\xi^4)\right]$$

So using standard trigonometric identities and results from Fourier series as given in equation (2-23), (4-4) and (4-6) we can express our solution as

$$\zeta'(Y,T) = Q''\xi[TY + (F_2(T+Y) - F_2(T-Y))/4] + O(\xi^3) \tag{2-20}$$

$$V'(Y,T) = Q''\xi c[(Y^2-1)/2 + (F_2(T+Y) + F_2(T-Y))/4] + O(\xi^3) \tag{2-21}$$

$$U'(Y,T) = Q''c[T(1+\xi^2(Y^2-1)/2) - \xi^2(F_3(T+Y) + F_3(T-Y))] + O(\xi^4) \tag{2-22}$$

In deriving this expression we need to know that the Fourier series of

$$F_3(x) = x(1-x^2/3) \; ; \quad -1 \leq x \leq 1 \; , \quad F_3(x+2) = F_3(x) \tag{2-23}$$

is

$$F_3(x) = \frac{4^3}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(2n-1)\pi x/2}{(2n-1)^4} \tag{2-24}$$

3. INTEGRAL TRANSFORM SOLUTION FOR A RECTANGULAR BASIN

3.1 Differential equation

If we substitute the expression $\zeta = \zeta' + \zeta^2$, $U = U' + U^2$, $V = V' + V^2$ into the differential equations (1-5), (1-6) and (1-7) and utilize the properties of ζ', U' and V' from section 2, we find that

$$\frac{\partial U^2}{\partial T} - \xi V^2 = c \frac{\partial \zeta^2}{\partial X}$$

$$\frac{\partial V^2}{\partial T} + \xi U^2 = c \frac{\partial \zeta^2}{\partial Y} \tag{3-1}$$

$$c \frac{\partial \zeta^2}{\partial T} = \frac{\partial U^2}{\partial X} + \frac{\partial V^2}{\partial Y}$$

We still desire a solution which starts from rest so

$$U^2(X,Y,0) = V^2(X,Y,0) = \zeta^2(X,Y,0) = 0 \tag{3-2}$$

To require that the tangential component of velocity, therefore transport also, vanishes at the edges of the basin yields

$$V^2(X,1,T) = V^2(X,-1,T) = 0 \tag{3-3}$$

and

$$U^2(0,Y,T) = U^2(L,Y,T) = -U'(Y,T), \; L = 2\ell/b \tag{3-4}$$

In order to uncouple equation (3-1) we apply the Laplace transform and obtain

$$s\overline{U}^2 - \xi\overline{V}^2 = c \frac{\partial \overline{\zeta}^2}{\partial X}$$

$$s\overline{V}^2 + \xi\overline{U}^2 = c \frac{\partial \overline{\zeta}^2}{\partial Y} \tag{3-5}$$

$$cs\overline{\zeta}^2 = \frac{\partial \overline{U}^2}{\partial X} + \frac{\partial \overline{V}^2}{\partial Y}$$

These equations may be combined to yield

$$(s^2+\xi^2)\overline{U}^2 = sc\,\frac{\partial\overline{\zeta}^2}{\partial X} + \xi c\,\frac{\partial\overline{\zeta}^2}{\partial Y}$$

$$(s^2+\xi^2)\overline{V}^2 = sc\,\frac{\partial\overline{\zeta}^2}{\partial Y} - \xi c\,\frac{\partial\overline{\zeta}^2}{\partial X} \tag{3-6}$$

$$\frac{\partial^2\overline{\zeta}^2}{\partial X^2} + \frac{\partial^2\overline{\zeta}^2}{\partial Y^2} - (s^2+\xi^2)\overline{\zeta}^2 = 0$$

The method of separation of variable is used to show that $\overline{\zeta}^2$, $\overline{V}^2$ and $\overline{U}^2$ can be written as

$$\begin{aligned}\overline{\zeta}^2(X,Y,s) &= A_0e^{-\xi Y-sX} + B_0e^{\xi Y+sX}\\ &+ c\sum_{n=}^{\infty} A_n[s\lambda_nc_n(Y) + \xi\beta'_ns_n(Y)]e^{-\beta'_nX}\\ &+ c\sum_{n=1}^{\infty} B_n[s\lambda_nc_n(Y) - \xi\beta'_ns_n(Y)]\,e^{\beta'_nX}\end{aligned} \tag{3-7}$$

$$\overline{V}^2(X,Y,s) = c^2\sum_{n=1}^{\infty} A_n(\lambda_n{}^2+\xi^2)s_n(Y)e^{-\beta'_nX} + c^2\sum_{n=1}^{\infty} B_n(\lambda_n{}^2+\xi^2)s_n(Y)e^{\beta'_nX} \tag{3-8}$$

$$\begin{aligned}\overline{U}^2(X,Y,s) &= -cA_0e^{-\xi Y-sX} + cB_0e^{\xi Y+sX} - c^2\sum_{n=1}^{\infty} A_n[\lambda_n\beta'_nc_n(Y)+s\xi s_n(Y)]e^{-\beta'_nX}\\ &+ c^2\sum_{n=1}^{\infty} B_n[\lambda_n\beta'_nc_n(Y)-s\xi s_n(Y)]e^{\beta'_nX}\end{aligned} \tag{3-9}$$

In the above equation $\lambda_n = n\pi/2$, $n = 1,2,3,\cdots$, $\beta'_n = \sqrt{\lambda_n^2 + s^2 + \xi^2}$, $c_n(Y) = \cos\lambda_n(1-Y)$, $s_n(Y) = \sin\lambda_n(1-Y)$ and arbitrary constants A_n, B_n, $n = 0,1,2,\cdots$, are still to be determined.

3.2 Perturbation technique

The last condition to be satisfied is that $\overline{U}^2 = -\overline{U}'$ at $X = 0,L$. This is difficult since the functions multiplying the arbitrary constants in equation (3-9) do not form an orthogonal set. We therefore limit ourselves to long narrow channels where the parameter $\xi = bf/2c$ (the ratio of the halfwidth of the basin to the internal radius of deformation) is less than 1. Then we expand both $\overline{U}^2$ and $-\overline{U}'$ in a power series in ξ and determine the arbitrary constants uniquelly by specifying the equation be true for each power of ξ. To this end we need the following expansions

$$A_n = A_n^{(0)} + A_n^{(1)}\xi + A_n^{(2)}\xi^2 + \cdots = \sum_{m=0}^{\infty} A_n^{(m)}\xi^m \qquad n = 0,1,2,\cdots$$

$$B_n = B_n^{(0)} + B_n^{(1)}\xi + B_n^{(2)}\xi^2 + \cdots = \sum_{m=0}^{\infty} B_n^{(m)}\xi^m \qquad n = 0,1,2,\cdots$$

$$e^{\xi Y} = \sum_{m=0}^{\infty} (\xi Y)^m/m!$$

$$\beta_n' = \beta_n\left(1+ \frac{\xi^2}{2(s^2+\lambda_n^2)} + 0(\xi^4)\right), \qquad \beta_n = \sqrt{\lambda_n^2 + s^2}$$

$$\lambda_n\beta_n' c_n(Y) \pm s\xi s_n(Y) = \beta_n\lambda_n c_n(Y) \pm s\xi s_n(Y) + \frac{\lambda_n\xi^2}{2\beta_n} c_n(Y) + 0(\xi^4)$$

$$\overline{U}'(Y,s) = \frac{Q}{s} + \frac{Q}{s^3}\left[-1 + \frac{\cosh sY}{\cosh s}\right]\xi^2 + 0(\xi^4).$$

We first list the equationsresulting from this expansion for powers of ξ^0 and ξ^1 as

$$\xi^0: \quad c(B_0^{(0)} - A_0^{(0)}) + c^2 \sum_{m=1}^{\infty} (B_m^{(0)} - A_m^{(0)})\lambda_m\beta_m c_m(Y) = - Q/s \tag{3-10}$$

$$c(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL}) + c^2 \sum_{m=1}^{\infty} (B_m^{(0)}e^{\beta_m L} - A_m^{(0)}e^{-\beta_m L})\lambda_m\beta_m c_m(Y) = - Q/s \tag{3-11}$$

$$\xi^1: \quad c(B_0^{(1)} - A_0^{(1)}) + cY(B_0^{(0)} + A_0^{(0)}) + c^2 \sum_{n=1}^{\infty} (B_m^{(1)} - A_m^{(1)})\lambda_m\beta_m c_m(Y)$$

$$- c^2 s \sum_{m=1}^{\infty} (B_m^{(0)} + A_m^{(0)}) s_m(Y) = 0 \tag{3-12}$$

$$c(B_0^{(1)}e^{sL} - A_0^{(1)}e^{-sL}) + cY(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL})$$

$$+ c^2 \sum_{m=1}^{\infty} (B_m^{(1)} e^{\beta_m L} - A_m^{(1)}e^{-\beta_m L})\lambda_m\beta_m c_m(Y) - c^2 s \sum_{m=1}^{\infty} (B_m^{(0)}e^{\beta_m L} -$$

$$A_m^{(0)}e^{-\beta_m L}) s_m(Y) = 0 \tag{3-13}$$

Define the inner product $<\circ,\cdot>$ of two functions by

$$<f,g> = \int_{-1}^{1} f(Y)g(Y)\, dY.$$

If we now take the inner product of equation (3-10) and (3-11) with 1 and use the fact that $<1,c_n(Y)> = 0$ we obtain

$$2c(B_0^{(0)} - A_0^{(0)}) = -2Q/s \,, \quad 2c(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL}) = -2Q/s$$

with solution

$$A_0^{(0)} = \frac{Q(e^{sL}-1)}{cs(e^{sL}-e^{-sL})} \;; \quad B_0^{(0)} = \frac{Q(e^{-sL}-1)}{cs(e^{sL}-e^{-sL})} \tag{3-14}$$

If we take the inner product of equations (3-10) and (3-11) with $c_n(Y)$ and use the fact that $\langle c_n(Y), c_m(Y)\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$ we get

$$c^2(B_n^{(0)} - A_n^{(0)})\lambda_n\beta_n = 0 \;; \quad c^2(B_n^{(0)}e^{\beta_n L} - A_n^{(0)}e^{-\beta_n L}) = 0$$

with solution

$$A_n^{(0)} = B_n^{(0)} = 0 \tag{3-15}$$

Proceeding to the terms multiplying ξ^1 (i.e. equations (3-12) and (3-13)) and taking the inner product with 1 yields

$$2c(B_0^{(1)} - A_0^{(1)}) = 0 \;; \; 2c(B_0^{(1)}e^{sL} - A_0^{(1)}e^{-sL}) = 0$$

with solution

$$A_0^{(1)} = B_0^{(1)} = 0 \tag{3-16}$$

We now make use of (3-15) and (3-16) to simplify equations (3-12) and (3-13) before taking the inner product with $c_n(Y)$ and obtain

$$c^2(B_n^{(1)} - A_n^{(1)})\lambda_n\beta_n + c(B_0^{(0)} + A_0^{(0)})\langle Y, c_n(Y)\rangle = 0$$

$$c^2(B_n^{(1)}e^{\beta_n L} - A_n^{(1)}e^{-\beta_n L})\lambda_n\beta_n + c(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL})\langle Y, c_n(Y)\rangle = 0$$

These may be solved and yield

$$A_n^{(1)} = \frac{Q\langle Y, c_n(Y)\rangle}{c^2 s \lambda_n \beta_n}\left(\frac{1-e^{-sL}}{1+e^{-sL}}\right)\left(\frac{1}{1-e^{-\beta_n L}}\right) \tag{3-17}$$

$$B_n^{(1)} = \frac{Q\langle Y, c_n(Y)\rangle}{c^2 s \lambda_n \beta_n}\left(\frac{1-e^{-sL}}{1+e^{-sL}}\right)\left(\frac{e^{-\beta_n L}}{1-e^{-\beta_n L}}\right) \tag{3-18}$$

The fact that $A_n^{(0)} = B_n^{(0)} = A_0^{(1)} = B_0^{(1)} = 0$ simplifies the expressions we obtain for the ξ^2 terms from the boundary condition which we now write as

$$\xi^2: \quad c(B_0^{(2)}-A_0^{(2)}) + cY^2(B_0^{(0)}-A_0^{(0)})/2 + c^2\sum_{m=1}^{\infty}(B_m^{(2)}-A_m^{(2)})\lambda_m\beta_m\, c_m(Y)$$

$$- c^2 s\sum_{m=1}^{\infty}(B_m^{(1)}+A_m^{(1)})s_n(Y) = \frac{Q}{s^3}\left(1-\frac{\cosh sY}{\cosh s}\right)$$

$$c(B_0^{(2)}e^{sL} - A_0^{(2)}e^{-sL}) + cY^2(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL})/2$$

$$+ c^2\sum_{m=1}^{\infty}(B_m^{(2)}e^{\beta_m L} - A_m^{(2)}e^{-\beta_m L})\lambda_m\beta_m c_m(Y)$$

$$- c^2 s\sum_{m=1}^{\infty}(B_m^{(1)}e^{\beta_m L} + A_m^{(1)}e^{-\beta_m L})s_m(Y) = \frac{Q}{s^3}\left(1-\frac{\cosh sY}{\cosh s}\right)$$

Now we take the inner product of these two equations with 1 to obtain

$$2c(B_0^{(2)}-A^{(2)}) + \frac{c}{3}(A_0^{(0)}+B_0^{(0)}) - c^2 s \sum_{m=1}^{\infty} A_m^{(1)}+B_m^{(1)})<1,s_m(Y)> =$$

$$- \frac{2Q}{s^3}(-1 + \frac{\tanh s}{s})$$

$$2c(B_0^{(2)}e^{sL}-A_0^{(2)}e^{-sL}) + \frac{c}{3}(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL})$$

$$- c^2 s \sum_{m=1}^{\infty} (B_m^{(1)}e^{\beta_m L} + A_m^{(1)}e^{-\beta_m L})<1,s_m(Y)> = \frac{-2Q}{s^3}(-1 + \frac{\tanh s}{s})$$

These may be solved to yield

$$A_0^{(2)} = \frac{Q}{cs}\left[\frac{\tanh s - s}{s^3} - \frac{1}{6}\right]\frac{1}{1+e^{-sL}}$$

$$- \frac{Q}{2c}\frac{(1-e^{-sL})}{(1+e^{-sL})^2}\sum_{m=1}^{\infty}\frac{<1,s_n(Y)><Y,c_m(Y)>}{\lambda_m\beta_m}\frac{(1+e^{-\beta_m L})}{(1-e^{-\beta_m L})} \tag{3-19}$$

$$B_0^{(2)} = - \frac{Q}{cs}\left[\frac{\tanh s - s}{s^3} - \frac{1}{6}\right]\frac{e^{-sL}}{1+e^{-sL}}$$

$$+ \frac{Q}{2c}\frac{(1-e^{-sL})}{(1+e^{-sL})^2}e^{-sL}\sum_{m=1}^{\infty}\frac{<1,s_m(Y)><Y,c_m(Y)>}{\lambda_m\beta_m}\frac{(1+e^{-\beta_m L})}{(1-e^{-\beta_m L})} \tag{3-20}$$

Finally we take the inner product of the ξ^2 equation with $c_n(Y)$ to obtain

$$c^2(B_n^{(2)}-A_n^{(2)})\lambda_n\beta_n + c^2(B_0^{(0)}-A_0^{(0)})<Y^2,c_n(Y)>/2$$

$$- c^2 s \sum_{m=1}^{\infty}(B_m^{(1)}+A_m^{(1)})<s_m(Y),c_n(Y)> = - \frac{Q<\cosh s\, Y,c_n(Y)>}{s^3\cosh s}$$

$$c^2(B_n^{(2)}e^{\beta_n L}-A_n^{(2)}e^{-\beta_n L})\lambda_n\beta_n + c^2(B_0^{(0)}e^{sL} - A_0^{(0)}e^{-sL})<Y^2,c_n(Y)>/2$$

$$-c^2 s\sum_{m=1}^{\infty}(B_m^{(1)}e^{\beta_m L}-A_m^{(1)}e^{-\beta_m L})<s_m(Y),c_n(Y)> = \frac{-Q<\cosh s\,Y,c_n(Y)>}{s^3\cosh s}$$

with solution

$$A_n^{(2)} = \frac{Q}{c^2 s\lambda_n\beta_n}\left\{\left(\frac{<\cosh s\,Y,c_n(Y)>}{s^2\cosh s} - \tfrac{1}{2}<Y^2,c_n(Y)>\right)\frac{1}{(1+e^{-\beta_n L})}\right.$$
$$\left.-s\left(\frac{1-e^{-sL}}{1+e^{-sL}}\right)\sum_{m=1}^{\infty}\frac{<s_m(Y),c_n(Y)><Y,c_m(Y)>(1+e^{-\beta_m L})}{\lambda_m\beta_m(1+e^{-\beta_n L})(1-e^{-\beta_m L})}\right\} \tag{3-21}$$

$$B_n^{(2)} = \frac{Q}{c^2 s\lambda_n\beta_n}\left\{\left(\frac{-<\cosh s\,Y,c_n(Y)>}{s^2\cosh s} + \tfrac{1}{2}<Y^2,c_n(Y)>\right)\left(\frac{e^{-\beta_n L}}{1+e^{-\beta_n L}}\right)\right.$$

$$\left.+s\left(\frac{1-e^{-sL}}{1+e^{-sL}}\right)\sum_{m=1}^{\infty}\frac{<s_m(Y),c_n(Y)><Y,c_m(Y)>(1+e^{-\beta_m L})}{\lambda_m\beta_m\,(1-e^{-\beta_m L})}\;\frac{e^{-\beta_n L}}{(1+e^{-\beta_n L})}\right. \tag{3-22}$$

In the above expressions we can determine that

$$<Y^2,c_n(Y)> = \frac{2}{\lambda_n^{2}}(1+\cos n\pi) \quad ; \quad <Y,c_n(Y)> = \frac{1}{\lambda_n^{2}}(1-\cos n\pi)$$

$$<1,s_n(Y)> = \frac{1}{\lambda_n}(1-\cos n\pi) \quad ; \quad <\cosh s Y c_n(Y)> = \frac{s\sinh s}{s^2+\lambda_n^{2}}(1+\cos n\pi)$$

$$<s_m(Y),c_n(Y)>=\tfrac{1}{2}\left[\frac{1-\cos(m-n)\pi}{\lambda_m-\lambda_n} + \frac{1-\cos(m+n)\pi}{\lambda_m+\lambda_n}\right] \tag{3-23}$$

3.3 Expression for $\bar{\zeta}(X,Y,s)$

We now summarize our results in terms of the expression for $\bar{\zeta}(X,Y,s) = \bar{\zeta}'(X,Y,s) + \bar{\zeta}^2(X,Y,s)$:

$$\bar{\zeta}'(X,Y,s) = \frac{Q\xi}{cs\kappa}\,\frac{\sinh \kappa Y}{\cosh \kappa} \quad , \qquad \kappa^2 = s^2 + \xi^2$$

$$\begin{aligned}\bar{\zeta}^2(X,Y,s) = {} & (A_0^{(0)} + \xi^2 A_0^{(2)})e^{-\xi Y - sX} + (B_0^{(0)} + \xi^2 B_0^{(2)})e^{\xi Y + sX} \\ & + c\sum_{n=1}^{\infty}(\xi A_n^{(1)} + \xi^2 A_n^{(2)})(s\lambda_n c_n(Y) + \xi\beta_n' s_n(Y))e^{-\beta_n' X} \\ & + c\sum_{n=1}^{\infty}(\xi B_n^{(1)} + \xi^2 B_n^{(2)})(s\lambda_n c_n(Y) - \xi\beta_n' s_n(Y)) e^{\beta_n' X}\end{aligned} \tag{3-24}$$

when $\lambda_n = n\pi/2$, $\beta_n' = \sqrt{\lambda_n^2 + s^2 + \xi^2}$ and the $A_n^{(m)}$ & $B_n^{(m)}$ are given by equations (3-14) through (3-22). At this point we have satisfied all conditions exactly except for the one for zero flux at the two "short" ends of the basin ($X = 0$ & $X = L$) which we have satisfied to order ξ^2. If we now expand the functions in equation (3-24) in terms of ξ we can write $\bar{\zeta}^2(X,Y,s)$ as

$$\bar{\xi}^2(X,Y,s) = \frac{Q}{cs\cosh sL/2}\Big[(1+\xi^2Y^2/2)\sinh s(L/2-X) - \xi Y\cosh s(L/2-X)$$

$$+\xi^2\sinh s\,(L/2-X)\left\{\frac{\tanh s - s}{s^3} - \frac{1}{6} - \frac{s\tanh sL/2}{2}\sum_{n=1}^{\infty}\right.$$

$$\left.\frac{<1,s_n(Y)><Y,c_n(Y)>}{\lambda_n\beta_n}\coth\beta_n' L/2\right\}\Big]$$

$$+\frac{\xi Q}{c}\tanh sL/2\sum_{n=1}^{\infty}\frac{<Y,c_n(Y)>}{\beta_n}\;\frac{\cosh\beta_n(L/2-X)}{\sinh\beta_n L/2}\;c_n(Y)$$

$$+\frac{\xi^2 Q}{cs}\Big[\tanh sL/2\sum_{n=1}^{\infty}\frac{<Y,c_n(Y)>}{\lambda_n}\;\frac{\sinh\beta_n(L/2-X)}{\sinh\beta_n L/2}\;s_n(Y)$$

$$+s\sum_{n=1}^{\infty}\left(\frac{<\cosh sY,c_n(Y)>}{s^2\cosh s} - \tfrac{1}{2}<Y^2,c_n(Y)>\right)\frac{1}{\beta_n}\;\frac{\sinh\beta_n(L/2-X)}{\cosh\beta_n L/2}\;c_n(Y)$$

$$-s^2\tanh sL/2\sum_{n=1}^{\infty}\;\sum_{m=1}^{\infty}\frac{<s_m(Y),c_n(Y)><Y,c_m(Y)>}{\lambda_m\beta_m\beta_n}$$

$$\coth\beta_m L/2\;\frac{\sinh\beta_n(L/2-X)}{\cosh\beta_n L/2}\;c_n(Y)\Big] \qquad (3\text{-}25)$$

The inner products occurring above may be evaluated as

$$<Y^2,c_n(Y)> = 2(1+\cos n\pi)/\lambda_n^2\;; \qquad <Y,c_n(Y)> = (1-\cos n\pi)/\lambda_n^2$$

$$<1,s_n(Y)> = (1-\cos n\pi)/\lambda_n\;; \qquad <\cosh s\,Y,c_n(Y)> = \frac{s\sinh s}{s^2+\lambda_n^2}(1+\cos n\pi)$$

$$<s_m(Y),c_n(Y)> = \tfrac{1}{2}\left[\frac{1-\cos(m+n)\pi}{\lambda_m+\lambda_n} + \frac{1-\cos(m-n)\pi}{\lambda_m-\lambda_m}\right]$$

Since the value of $\bar{\tau}_x$ (the forcing function) appears in the expression for Q, it must be specified before the inverse transform is to be found. In section 4 this will be done for a wind which rises quickly from rest to a steady value. However, regardless of the form of the forcing function, it is clear from the form of (3-25) that $\bar{\zeta}^2(X+L/2,Y,s) = -\bar{\zeta}^2(X-L/2,-Y,s)$ in the solution is asymmetric about the center point of the basin.

3.4 Limiting values for large L (a semi-infinite basin).

The expression for $\bar{\zeta}^2$ is somewhat simplified if we consider a very long basin $(L \to \infty)$. In deriving this following expression we use the limits given in table 2.

$f(L)$	$\lim_{L\to\infty} f(L)$
$\dfrac{\sinh\beta_n(L/2-X)}{\sinh\beta_n L/2}$	$e^{-\beta_n X}$
$\dfrac{\sinh\beta_n(L/2-X)}{\cosh\beta_n L/2}$	$e^{-\beta_n X}$
$\dfrac{\cosh\beta_n(L/2-X)}{\sinh\beta_n L/2}$	$e^{-\beta_n X}$
$\coth \beta_n L/2$	1
$\dfrac{\sinh s(L/2-X)}{\cosh s L/2}$	e^{-sX}
$\dfrac{\cosh s(L/2-X)}{\cosh s L/2}$	e^{-sX}
$\tanh s L/2$	1

Table 2: Limits as L approaches infinity.

$$
\begin{aligned}
\bar{\zeta}^2_{L\to\infty}(X,Y,s) = \frac{Q}{cs}\Bigg\{\Bigg[1 - \xi Y + \xi^2\Bigg(Y^2/2 + \frac{\tanh s - s}{s^3} - \frac{1}{6} \\
- \frac{s}{2}\sum_{n=1}^{\infty}\Big(\frac{1-\cos n\pi}{\lambda_n^2}\Big)^2\frac{1}{\beta_n}\Bigg)\Bigg]e^{-sX} \\
+ \xi s \sum_{n=1}^{\infty}\frac{1-\cos n\pi}{\lambda_n^2\beta_n}\, e^{-\beta_n X}\, c_n(Y) \\
+ \xi^2 \sum_{n=0}^{\infty}\frac{1-\cos n\pi}{\lambda_n^3}\, e^{-\beta_n X}\, s_n(Y) \\
+ \xi^2 s \sum_{n=1}^{\infty}\Bigg(\frac{\tanh s}{s\beta_n^2} - \frac{1}{\lambda_n^2}\Bigg)(1+\cos n\pi)\frac{e^{-\beta_n X}}{\beta_n}\, c_n(Y) \\
- \xi^2 s^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\langle s_m, c_n\rangle(1-\cos m\pi)}{\lambda_m^3\beta_m\beta_n}\, e^{-\beta_n X}\, c_n(Y)\Bigg\} \qquad (3\text{-}26)
\end{aligned}
$$

4 INVERSION FOR A WIND STRESS = $\tau_0 H(T)$

4.1 Rectangular Basin

For a wind stress which suddenly rises from rest to a steady value we take $\tau_x = \tau_0 H(T)$, $H(T)$ the Heaviside function, and calculate its Laplace transform as $\bar{\tau}_x = \tau_0/s$. Thus the value of Q/cs, which occurs in many equations from (2-5) through (3-26) becomes

$$\frac{Q}{cs} = \frac{b\ \tau_0}{2c^2\rho(1+h/H)}\left(\frac{1}{s^2}\right) = \frac{Q''}{s^2} \tag{4-1}$$

If we make the replacement in equation (3-25) we can proceed to find the inverse Laplace transform of $\bar{\zeta}(X,Y,s)$. Unfortunately not all the expressions occurring in this equation have inverse Laplace transforms listed in the standard table (for example Roberts-Kaufman (1966) or Erdelyi et.al. (1954)). It turns out to be very useful to consider the fact that L is relatively large and expand some of the functions of L in term of powers of e^{-sL} and $e^{-\beta_n L}$ as is appropriate. These expansions appear in Table 3 below and are used to rewrite the already lengthy expression for $\bar{\zeta}^2(X,Y,s)$. Details for the solution up to order ξ^2 follow. The list of Laplace transforms used in these caluclations appear in Appendix A.

We rewrite the first, second and fourth term from equation (3-25)

$$\bar{\zeta}^2(X,Y,s) = Q''\left\{(1+\xi^2Y^2/2\ \frac{\sinh s\,(L/2-X)}{s^2\cosh s\,L/2} - \xi Y\ \frac{\cosh s\,(L/2-X)}{s^2\cosh s\,L/2}\right.$$

$$\left. + \xi\ \frac{\tanh s\,L/2}{s}\sum_{n=1}^{\infty}\frac{\langle Y,c_n(Y)\rangle}{\beta_n}\ \frac{\cosh \beta_n(L/2-X)}{\sinh \beta_n L/2}\ c_n(Y)\right\} \tag{4-2}$$

	Function	Expansion
1	$\tanh s L/2$	$\sum_{h=0}^{\infty} a_h (-1)^h e^{-hsL}, a_0 = 1,\ a_h = 2,\ h \geq 1$
2	$\coth \beta_n L/2$	$\sum_{h=0}^{\infty} a_h e^{-\beta_n hL},\ a_0 = 1,\ a_h = 2,\ h \geq 1$
3	$\dfrac{\cosh \beta_n (L/2-X)}{\sinh \beta_n L/2}$	$\sum_{h=0}^{\infty} \left\{ e^{-\beta_n (X+hL)} + e^{-\beta_n ((h+1)L-X)} \right\}$
4	$\dfrac{\cosh s\,(L/2-X)}{\cosh s\,L/2}$	$\sum_{h=0}^{\infty} \left\{ (-1)^h \right\} \left\{ e^{-s(X+hL)} + e^{-s((h+1)L-X)} \right\}$
5	$\dfrac{\sinh \beta_n (L/2-X)}{\sinh \beta_n (L/2}$	$\sum_{h=0}^{\infty} \left\{ e^{-\beta_n (X+hL)} - e^{-\beta_n ((h+1)L-X)} \right\}$
6	$\dfrac{\sinh \psi (L/2-X)}{\cosh \psi L/2}$	$\sum_{h=0}^{\infty} (-1)^h \left\{ e^{-\psi (X+hL)} - e^{-\psi ((h+1)L-X)} \right\}$

ψ = either s or β_n

Table 3. Expansion in terms of exponentials

The inverse Laplace transforms of the first two terms are given in Appendix A, but the expansion from table 3 must be used to rewrite the last term in equation (4-2) as

$$\frac{Q''\xi}{s}\sum_{k=0}^{\infty} a_k(-1)^k e^{-ksL}\sum_{n=0}^{\infty}\frac{<Y,c_n(Y)>c_n(Y)}{\beta_n}\sum_{m=0}^{\infty}\left\{e^{-\beta_n(X+mL)}+e^{-\beta_n((m+1)L-X)}\right\}$$

We now see that each term above has the form

$\dfrac{e^{-ksL}}{s\beta_n}\, e^{-\beta_n\psi}$ and can be inverted using the fifth function from Appendix A. We collect these results (together with the expansion to order ξ^2 of $\zeta'(Y,T)$ from equation (2-17)) as

$$\zeta(X,Y,T) = Q''\Bigg[\left\{L/2-X-\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi X/L\ \cos(2n-1)\pi T/L}{(2n-1)^2}\right\}$$

$$\left\{1+\xi^2Y^2/2\right\}+\xi\left\{\frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(2n-1)\pi Y/2\ \sin(2n-1)\pi T/2}{(2n-1)^3}\right.$$

$$\left.+\frac{4LY}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi X/L\ \sin(2n-1)\pi T/L}{(2n-1)^2}\right\}$$

$$+\xi\sum_{n=1}^{\infty}<Y,c_n(Y)>c_n(Y)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}a_k(-1)^k\left\{g_{1n}(T-kL,X+mL)+g_{1n}(T-kL,(m+1)L-X)\right\}\Bigg] \tag{4-3}$$

This solution may be put in a better form if we recall the Fourier series for two functions. One is the "sawtooth" function defined by $y = \begin{cases} x & 0 < x < \pi \\ 2\pi-x & \pi < x < 2\pi \end{cases}$ with a Fourier series $\dfrac{\pi}{2}-\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2}$.

If we now define a function $F_1(\psi)$ by

$$F_1(\psi) = \begin{cases} \psi & 0 < \psi < L \\ 2L-\psi, & L < \psi < 2L \end{cases} ; \quad F_1(\psi+2L) = F_1(\psi) \tag{4-4}$$

we discover its Fourier representation is

$$F_1(\psi) = \frac{L}{\pi}\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)(\psi\pi/L)}{(2n-1)^2}\right\} \tag{4-5}$$

Also we know that the Fourier expansion of the parabola

$$\left(\frac{\pi^2}{4} - x^2\right), \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \quad \text{is} \quad \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^3} .$$

If we define a function $F_2(\psi)$ by

$$F_2(\psi) = 1 - \psi^2 , \quad -1 \leq \psi \leq 1 ; \quad F_2(\psi+2) = F_2(\psi) \tag{4-6}$$

we have

$$F_2(\psi) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(2n-1)\pi\psi/2}{(2n-1)^3} \tag{4-7}$$

Use of trigonometric identities and the definition in equation (4-5) and (4-7) allow us to rewrite equation (4-3) as

$$\begin{aligned} \zeta(X,Y,T) = Q''\Big[& -X(1+\xi^2Y^2/2)+\tfrac{1}{2}(1+\xi Y+\xi^2Y^2/2)F_1(T+X) \\ & +\tfrac{1}{2}(1-\xi Y + \xi^2Y^2/2)F_1(T-X)+\xi\Big\{(F_2(T+Y)- F_2(T-Y))/4 \\ & +\sum_{n=1}^{\infty} \langle Y,c_n(Y)\rangle c_n(Y) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k(-1)^k \\ & \left\{g_{1n}(T-kL,X+mL)+g_{1n}(T-kL,(m+1)L-X)\right\}\Big\}\Big] \end{aligned} \tag{4-8}$$

In this form we see the solution consists of a static tilting (-X) plus "sawtooth" or Kelvin waves eminating from each end of the basin plus parabolic disturbances propagating from each of the lateral sides plus the last series (with its triple sum). Fortunately very few terms of this triple series are large enough to contribute significantly to the solution. If we use the result from Appendix B that

$$\int_X^T J_0(\lambda_1\sqrt{\tau^2-X^2})d\tau = \frac{2}{\pi}\left[e^{-\pi X/2} + \frac{2T^{-\frac{1}{2}}}{\pi}\sin\left(\frac{\pi T}{2} - \frac{\pi}{4}\right)\right]$$

for T large and consider the region $0 < X < L/2$ we can expand this triple sum with the results given in Table 4. Ignoring the terms for $n = 2$ and higher means that similar terms to those in Table 4 are omitted as they have multiplicative factors of $\frac{1}{27}$, $\frac{1}{125}$ etc. Also from Table 4 we can see that for T large the importance of these terms diminishes as X moves away from the end. For most problems $\pi L/2$ is much bigger than one so the terms inclosed in [ ] in Table 4 can be ignored and the expression for the series is somewhat simpler.

Using the series expansion no.6 from Table 3 allows us to write the first part of the term starting with $\xi^2\sinh s\,(L/2-X)$ in (3-25) as

$$\frac{Q''\xi^2}{s^2}\left\{\frac{\tanh s - s}{s^3} - \frac{1}{6}\right\}\left\{\sum_{h=0}^{\infty}(-1)^h[e^{-s(X+hL)} - e^{-s((h+1)L-X)}]\right\}$$

This can be inverted using inverse transforms number 3 and 7 from Appendix A as

$$Q''\xi^2\left\{\sum_{k=0}^{\infty}(-1)^k[\,g_2(X,T-kL)-g_2(-X,T-(k+1)L)] \right.$$
$$\left. -\tfrac{1}{2}[-X + \tfrac{1}{2}(F_1(T+X)+F_1(T-X))]\right\} \qquad (4-9)$$

where F_1 is the "sawtooth" function defined by equation (4-4).

Value of triple series	Interval
0	0<T<X
$\left\{\frac{16\xi}{\pi^3}\sin(\frac{\pi Y}{2})\right\}\left\{e^{-\pi X/2}+\frac{2T^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi T}{2}-\frac{\pi}{4})\right\}$	x<T<L-X
$\left\{\quad '' \quad\right\}\left\{e^{-\pi X/2}+\frac{4T^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi T}{2}-\frac{\pi}{4})+\left[e^{-\pi(L-X)/2}\right]\right\}$	L-X<T<L+X
$\left\{\quad '' \quad\right\}\left\{-e^{-\pi X/2}+\frac{6T^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi T}{2}-\frac{\pi}{4})-\frac{4(T-L)^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi(T-L)}{2}-\frac{\pi}{4})+\left[e^{-\pi(L-X)/2}+e^{-\pi(L+X)/2}\right]\right\}$	L+X<T<2L-X
$\left\{\quad '' \quad\right\}\left\{-e^{-\pi X/2}+\frac{8T^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi T}{2}-\frac{\pi}{4})-\frac{8(T-L)^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi(T-L)}{2}-\frac{\pi}{4})+\left[-e^{-\pi(L-X)/2}+e^{-\pi(L+X)/2}+e^{-\pi(2L-X)/2}\right]\right\}$	2L-X<T<2L+X
$\left\{\quad '' \quad\right\}\left\{e^{-\pi X/2}+\frac{10X^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi T}{2}-\frac{\pi}{4})-\frac{12(T-L)^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi(T-L)}{2}-\frac{\pi}{4})+\frac{4(T-2L)^{-\frac{1}{2}}}{\pi}\sin(\frac{\pi(T-2L)}{2}-\frac{\pi}{4})+\left[-e^{-\pi(L-X)/2}-e^{-\pi(L+X)/2}+e^{-\pi(2L-X)/2}+e^{-\pi(2L+X)/2}\right]\right\}$	2L+X<T<3L-X

Table 4 : Expansion of Triple Series

By using expansion 1, 2 and 6 from table 3 we can express the coefficient of $\frac{-Q''\xi^2}{2s}\frac{<1,s_n(Y)><Y,c_n(Y)>}{\lambda_n\beta_n}$ as

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_k a_m(-1)^m e^{-\beta_n kL-msL}\left(\sum_{j=0}^{\infty}(-1)^j e^{-s(X+jL)}-\sum_{j=0}^{\infty}(-1)^j e^{-s((j+1)L-X)}\right)$$

Each of these terms may be inverted using #5 of Appendix A and we finally obtain

$$\frac{-Q''\xi^2}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{<1,s_n(Y)><Y,c_n(Y)>}{\lambda_n}a_k a_m(-1)^{m+j}\Big\{g_{1n}(kL,T-X-(m+j)L)$$

$$-\ g_{1n}(kL,T+X-(m+j+1)L)\Big\} \tag{4-10}$$

The rest of the terms multiplying ξ^2 in equation (3-25) may be expanded by a similar procedure. Because of the length of the expression we omit the details and only record the final expression for $\zeta(X,Y,T)$ as

$$\zeta(X,Y,T) = Q''\Big[-X(1+\xi^2(Y^2-1)/2)+\tfrac{1}{2}(1+\xi Y+\xi^2(Y^2-1)/2)F_1(T+X)$$

$$+\tfrac{1}{2}(1-\xi Y+\xi^2(Y^2-1)/2)F_1(T-X)+\xi\Big\{(F_2(T+Y)-F_2(T-Y))/4$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1-\cos n\pi)}{\lambda n^2}c_n(Y)a_k(-1)^k\Big[g_{1n}(X+mL,T-kL)+g_{1n}((m+1)L-X,T-kL)\Big]\Big\}$$

$$+\xi^2\Big\{\sum_{k=0}^{\infty}(-1)^k(g_2(X,T-kL)-g_2(-X,T-(k+1)L))-\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(1-\cos n\pi)^2}{\lambda n^4}$$

$$a_k a_m(-1)^{m+j}\Big[g_{1n}(kL,T-X-(m+j)L)-g_{1n}(kL,T+X-(m+j+1)L)\Big]$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1-\cos n\pi)s_n(Y)}{\lambda n^3}a_m(-1)^m\Big[g_{7n}(X+kL,T-mL)-g_{7n}((k+1)L-X,T-mL)\Big]$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(1+\cos n\pi)c_n(Y)(-1)^k\Big[g_{6n}(X+kL,T)-g_{6n}((k+1)L-X,T)$$

$$-\{g_{1n}(X+kL,T)-g_{1n}((k+1)L-X,T)\}/\lambda_n^2\Big]$$

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{<s_m(Y),c_n(Y)><Y,c_m(Y)>c_n(Y)}{\lambda_m}a_j a_k(-1)^{j+k}$$

$$\cdot\Big[\alpha_{mn}(X+iL,jL,T-kL)-\alpha_{mn}((i+1)L-X,jL,T-kL)\Big]\Big\}\Big] \qquad (4\text{-}11)$$

4.2 Limiting values for large L (the semi-infinite basin)

Because L is a fairly large number it is useful to find a simpler expression for ζ in the limit as $L \to \infty$. This would especially be useful if concern was for times less than the time needed for the disturbance to travel from the other end. Fortunately the form of equation (4-11) is such that we can find this limit directly for most terms. Alternatively we can return to the transformed solution given by equation (3-26) and use results from Appendix A to find that for a step wind $(\tau_x = \tau_0 H(T))$

$$\zeta^2(X,Y,T) = Q''\Big[(1-\xi Y+\xi^2(Y^2-1)/2)f_1(X,T)+\xi\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)g_{1n}(X,T)c_n(Y)}{\lambda_n^2}$$

$$+\xi^2\Big\{g_2(X,T)-\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)^2}{\lambda_n^4}g_{1n}(0,T-X)+\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{\lambda_n^3}g_{7n}(X,T)s_n(Y)$$

$$+\sum_{n=1}^{\infty}(1+\cos n\pi)c_n(Y)[g_{6n}(X,T)-g_{1n}(X,T)/\lambda_n^2]$$

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{<s_m(Y),c_n(Y)><Y,c_m(Y)>c_n(Y)}{\lambda_m}\alpha_{mn}(X,0,T)\Big\}\Big] \qquad (4\text{-}12)$$

4.3 Special locations

The lengthy expression for $\zeta(X,Y,T)$ simplifies for certain locations along the basin, especially if we consider the solution up to terms of order ξ. For example if $X = Y = 0$, i.e. you consider motion at the midpoint of the downwind end, we find from equation (4-11) that

$$\zeta 0,0,T) = Q''F_1(T) \tag{4-13}$$

This shows that for ξ small, the motion at this point is represented by a "sawtooth" wave. In terms of the interpretation by Csanady (1975) this represents a "Kelvin" wave propagating back and forth along the channel.

If we now consider the solution at the sides of the channel, we find

$$\begin{aligned}\zeta(X,\pm 1,T) = Q''\Big[&-X+\{(1\pm\xi)F_1(T+X)+(1\mp\xi)F_1(T-X)\}/2\\ &+\xi\Big\{(F_2(T\pm 1)-F_2(T\mp 1))/4+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1-\cos n\pi)}{\lambda_n^2}c_n(\pm 1)a_k(-1)^k\\ &\Big[g_{1n}(X+mL,T-kL)+g_{1n}((m+1)L-X,T-kL)\Big]\Big\}\Big]\end{aligned} \tag{4-14}$$

Now use of the definition of F_2 from equation (4-7) allows us to combine the parabolic disturbance as

$$(F_2(T\pm 1) - F_2(T\mp 1))/4 = \frac{\pm 16}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi T/2}{(2n-1)^3}$$

Since the Fourier expansion of $x(\pi-x)$, $0 < x < \pi$, is $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$

we define a function $F_4(\psi)$ by

$$F_4(\psi) = \psi(1-\psi/2)\ , \qquad 0 \le \psi \le 2 \tag{4-15}$$

we have the Fourier series

$$F_4(\psi) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi\psi/2}{(2n-1)^3} \tag{4-16}$$

This gives the relationship that

$$(F_2(T\pm 1) - F_2(T\mp 1))/4 = F_4(T) \tag{4-17}$$

The last location where the solution simplifies greatly is at $X = L/2$. The best way to evaluate the solution here is to return to the transformed expression of equation (3-25) and see that

$$\overline{\zeta}^2(L/2,Y,s) = \frac{Q''}{s^2}\left[\frac{-\xi Y}{\cosh s\,L/2} + s\xi\tanh sL/2 \sum_{n=1}^{\infty} \frac{\langle Y,c_n(Y)\rangle c_n(Y)}{\beta_n \sinh \beta_n L/2} + O(\xi^3)\right] \tag{4-18}$$

Thus from Appendix A the first term above becomes

$$\xi Q''\left[-Y(T-\frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(2n-1)\pi\, T/L}{(2n-1)^2})\right]$$

Again we use Fourier series to simplify the form of the infinite series. Since the expansion of

$$x, \quad -\pi/2 < x \leq \pi/2 \quad \text{is} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(2n-1)x}{(2n-1)^2}$$

we define

$$F_5(\psi) = \psi, \qquad -L/2 < \psi < L/2, \qquad F_5(\psi+L) = F_5(\psi) \tag{4-19}$$

with Fourier expansion

$$F_5(\psi) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(2n-1)\,\pi T/L}{(2n-1)^2} \tag{4-20}$$

With this result we can express the first term as

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 \cosh s\,L/2}\right\} = T - F_5(T) \tag{4-21}$$

The second term in equation (4-18) must be expanded in powers of e^{-sL} and $e^{-\beta_n L}$ before inversion takes place. If we do so and combine the resulting expression to that of ζ' from equation (2-20) we finally obtain

$$\zeta(L/2,Y,T) = Q''\xi\Big[(F_2(T+Y) - F_2(T-Y))/4 + Y\,F_5(T) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \langle Y,c_n(Y)\rangle c_n(Y)a_k(-1)^k g_{1n}((m+\tfrac{1}{2})L,T-kL)\Big] + O(\xi^3) \tag{4-22}$$

5. COMPARISONS AND CONCLUSIONS

Since a journal article comparing the results from this linear analysis to a finite difference solution of the problem and with observational data is currently being prepared (with B. Gjevik and G. Mørk) this section is purposely quite brief. A natural question to ask is "Why do a perturbation analysis when it is also possible to do a finite difference approximation which doesn't require the assumption that ξ be less than 1." There are perhaps four answers to this question

1) It provides a check on the correctness of all the programming instructions needed for the finite difference solution.

2) It provides a solution where individual effects can be analyzed separately while the finite difference solution gives only the sum of all effects.

3) It provides qualitative features often difficult or impossible to determine from the finite difference solution.

4) It doesn't have the problem of errors accumulating and can be used even for large values of time.

Needless to say, the requirement that ξ be less than one is a limiting one. However, it appears from the finite difference solution that the periodic "sawtooth" at the end of the basin also exists for ξ greater than 1.

Figure 2 illustrates the comparison of this solution to order ξ with the finite difference solution of the linear set of equations given by (1-1) to (1-3). The curves are the finite difference solution while the circled points are from the perturbation analysis.

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APPENDIX A

List of inverse Laplace Transforms

	Function	Inverse transform	
1.	$\dfrac{\sinh s\,(L/2-X)}{s^2\cosh s\,L/2}$	$\dfrac{L}{2}-X-\dfrac{4L}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)\pi X/L\,\cos(2n-1)\pi T/L}{(2n-1)^2}$	
		(valid for $0 \le X \le L/2$)	
2.	$\dfrac{\cosh s\,(L/2-X)}{s^2\cosh s\,L/2}$	$T-\dfrac{4T}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)\pi X/L\,\sin(2n-1)\pi T/L}{(2n-1)^2}$	
3.	$\dfrac{e^{-sX}}{s^2}$	$f_1(X,T) = \begin{cases} 0 \\ T-X \end{cases}$	$0<T<X$ $X<T$
4.	$\dfrac{e^{-\sqrt{s^2+\lambda_n^2}\,X}}{\sqrt{s^2+\lambda_n^2}}$	$\begin{cases} 0 \\ J_0(\lambda_n\sqrt{T^2-X^2}) \end{cases}$	$0<T<X$ $X<T$
5.	$\dfrac{e^{-\sqrt{s^2+\lambda_n^2}\,X}}{s\sqrt{s^2+\lambda_n^2}}$	$g_{1n}(X,T) = \begin{cases} 0 \\ \int_X^T J_0(\lambda_n\sqrt{\tau^2-X^2})\,d\tau \end{cases}$	$0<T<X$ $X<T$
6.	$\dfrac{\tanh s}{s}$	$f_2(T) = \begin{cases} 1 \\ -1 \end{cases}$	$4n<T<4n+2$ $4n+2<T<4n+4$
7.	$\left\{\dfrac{\tanh s - s}{s^5}\right\} e^{-sX}$	$-\dfrac{1}{3}f_1(X,T)+g_2(X,T)$	$0<T<X$

$$g_2(X,T) = \begin{cases} 0 \\ \dfrac{2^6}{\pi^5}\sum_{m=1}^{\infty}\dfrac{\sin[\frac{2m-1}{2}\pi(T-X)]}{(2m-1)^5} & X<T \end{cases}$$

8. $$\frac{e^{-\sqrt{s^2+\lambda_n^2}X}}{s^2\sqrt{s^2+\lambda_n^2}} \qquad g_{3n}(X,T) = \begin{cases} 0 & 0 < T < X \\ \int_X^T (T-\tau)J_0(\lambda_n\sqrt{\tau^2-X^2})d\tau & X < T \end{cases}$$

9. $$\frac{e^{-\sqrt{s^2+\lambda_n^2}X}}{(s^2+\lambda_n^2)^{3/2}} \qquad g_{4n}(X,T) = \begin{cases} 0 & 0 < T < X \\ \frac{1}{\lambda_n}\int_X^T \sin\lambda_n(T-\tau) & X < T \\ \cdot J_0(\lambda_n\sqrt{\tau^2-X^2})d\tau & \end{cases}$$

10. $$\frac{\sqrt{s^2+\lambda_n^2}}{s^2} \qquad g_{5n}(T) = {}_1F_2(-\tfrac{1}{2};\tfrac{1}{2};1;-\lambda_n^2T^2/4)$$

$$= \sum_{m=0}^{\infty} \frac{(-\frac{1}{2})_m}{(\frac{1}{2})_m(1)_m m!}\left(\frac{-\lambda_n^2T^2}{4}\right)^m$$

11. $$\frac{\tanh s}{s}\,\frac{e^{-\sqrt{\lambda_n^2+s^2}X}}{(s^2+\lambda_n^2)^{3/2}} \qquad g_{6n}(X,T) = \begin{cases} 0 & 0 < T < X \\ \int_X^T g_{4n}(X,T-\tau)f_2(\tau)d\tau & X < T \end{cases}$$

12. $$\frac{e^{-\sqrt{s^2+\lambda_n^2}X}}{s^2} \qquad g_{7n}(X,T) = \begin{cases} 0 & 0 < T < X \\ \int_X^T g_{5n}(T-\tau)J_0(\lambda_n\sqrt{\tau^2-X^2})d\tau & X < T \end{cases}$$

13. $$\frac{e^{-\sqrt{\lambda_n^2+s^2}X_1}\;e^{-\sqrt{\lambda_m^2+s^2}X_2}}{(s^2+\lambda_n^2)^{\frac{1}{2}}(s^2+\lambda_m^2)^{\frac{1}{2}}} \qquad \alpha_{mn}(X_1,X_2,T) = \begin{cases} 0 & 0 < T < X \\ \int_{X_1}^{T-X_2} J_0(\lambda_n\sqrt{(T-\tau)^2-X_2^2}) & \\ \cdot J_0(\lambda_m\sqrt{\tau^2-X_1^2})d\tau & X < T \end{cases}$$

APPENDIX B

Asymptotic evaluation of $\int_X^T J_0(\lambda_n\sqrt{\tau^2 - X^2})d\tau$

The simplest way of evaluating the behavior of the above integral seems to be by use of Laplace theory. We know from 5. Appendix A that

$$\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s^2+\lambda_n^2}X}}{s\sqrt{s^2+\lambda_n^2}}\right\} = \begin{cases} 0 & 0 < T < X \\ \int_X^T J_0(\lambda_n\sqrt{\tau^2-X^2})d\tau & X < T \end{cases}$$

To use the theorem on asymptotic evaluation of functions as found in Doetsch (1971), pages 150 and 151, we note that the above function of s is not analytic at $s = 0$, $\pm i\lambda_n$, all with the same real part. This means we need to find the contribution from all these points.

At $s = 0$ we can expand the function by rewriting it as

$$\frac{1}{\lambda_n s}(1+(s/\lambda_n)^2)^{-\frac{1}{2}}\, e^{\lambda_n X(1+(s/\lambda_n)^2)^{\frac{1}{2}}}$$

$$= \frac{e^{-\lambda_n X}}{\lambda_n s} + \text{ a power series in } s$$

Therefore from this pole we find a contribution to the asymptotic behavior of the integral as $\lambda_n^{-1}\, e^{-\lambda_n X}$.

At the points $s = \pm i\lambda_n$ we write the function of s as

$$\frac{\pm 1}{i\lambda_n\sqrt{\pm 2i\lambda_n}}\left(\frac{1}{\sqrt{s\mp i\lambda_n}} - X \pm 2i\lambda_n(s\mp i\lambda_n)^{\frac{1}{2}}\, X^2/2 + \ldots\right)$$

yielding the asymptotic expression

$$e^{\pm i\lambda_n T}\left(\frac{\pm 1}{i\lambda_n\sqrt{\mp 2i\lambda_n}}\, T^{-\frac{1}{2}} + (\quad)\, T^{-\frac{3}{2}} + \ldots\right)$$

Combining the above results we have

$$\int_X^T J_0(\lambda_n\sqrt{\tau^2-X^2})d\tau = \frac{e^{-\lambda_n X}}{\lambda_n} + \frac{2\sin(\lambda_n T - \pi/4)}{\lambda_n\sqrt{2\lambda_n}\,\Gamma(\frac{1}{2})}\, T^{-\frac{1}{2}}$$

For the lowest eigenvalue this becomes

$$\int_X^T J_0(\lambda_1\sqrt{\tau^2-X^2})d\tau = \frac{2}{\pi}\left[e^{-\pi X/2} + \frac{2T^{-\frac{1}{2}}\sin(\pi T/2 - \pi/4)}{\pi}\right] .$$